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Conditions for an Extremum in Metric Spaces*

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Abstract. General necessary and sufficient conditions of the k-th order (where $k \geqslant 0$) for an extremum of an arbitrary function defined on an arbitrary metric space are stated. Examples illustrating the theory are described.

Key words: Metric space; Local (global) minimum and maximum; k-th order necessary optimality condition; k-th order sufficient condition; k-th order rate of steepest descent and ascent

1. Introduction

Let X be a metric space with the metric ρ and let us assume that a functional f is defined on X and takes values from $\overline{\mathbb{R}} = [-\infty, +\infty]$. Our aim is to describe k-th order necessary and sufficient conditions for a maximum and a minimum of f on X. It turns out that such conditions can be formulated in a very general form by means of the k-th order rates of steepest descent and ascent introduced in the paper. Being applied to specific spaces (normed spaces, for example) the obtained conditions generate, among others, some well-known optimality conditions (see, e.g., [1–6]).

In Section 2 the first-order conditions are proved. In Section 3 the k-th order conditions are described.

2. First-order optimality conditions

Let X be a metric space with the metric ρ and let us assume that a functional f is defined on X and takes values from $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\} = [-\infty, +\infty]$.

Put

$$dom\ f = \{x \in X \mid f(x) \in \mathbb{R}\}$$

and assume that

$$dom\ f \neq \emptyset\,. \tag{2.1}$$

Let $x \in dom\ f$. Denote

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$$f^{\downarrow}(x) = \liminf_{\substack{y\in X \\ y\to x}} \frac{f(y)-f(x)}{\rho(x,y)} . \tag{2.2}$$

If there exists no sequence $\{y_k\}$, such that

$$y_k \in X,\ y_k \neq x \forall k , \quad y_k \to x ,$$

then by definition $f^{\downarrow}(x) = +\infty$. Since $x \in dom\ f$, then the limit in (2.2) always exists though it may be equal to $+\infty$ or $-\infty$).

The quantity $f^{\downarrow}(x)$ is called the *rate of steepest descent* of the function f at the point x.

(2.2) implies the expansion

$$f(y) = f(x) + \rho(x,y) f^{\downarrow}(x) + \underline{o}(\rho(x,y)) ,$$

where

$$\liminf_{y\to x} \frac{\underline{o}(\rho(x,y))}{\rho(x,y)} = 0 . \tag{2.3}$$

Analogously, for $x \in dom\ f$ one can define the quantity

$$f^{\uparrow}(x) = \limsup_{\substack{y\in X \\ y\to x}} \frac{f(y)-f(x)}{\rho(x,y)} . \tag{2.4}$$

If there exists no sequence $\{y_k\}$, such that

$$y_k \in X,\ y_k \neq x \forall k , \quad y_k \to x ,$$

then by definition $f^{\uparrow}(x) = -\infty$. Since $x \in dom\ f$, then the limit in (2.4) always exists though it may be equal to $+\infty$ or $-\infty$).

(2.4) implies the expansion

$$f(y) = f(x) + \rho(x,y) f^{\uparrow}(x) + \overline{o}(\rho(x,y)) ,$$

where

$$\limsup_{y\to x} \frac{\overline{o}(\rho(x,y))}{\rho(x,y)} = 0 . \tag{2.5}$$

The quantity $f^{\uparrow}(x)$ is called the *rate of steepest ascent* of the function f at the point x.

Put

$$f_* = \inf_{x\in X} f(x) , \quad f^* = \sup_{x\in X} f(x) .$$

It follows from (2.1) that

$$f_* < +\infty , \quad f^* > -\infty .$$

If for some point $x_* \in X$ it holds that $f(x_*) = f_*$, then the point x_* is called a *minimum point* (or a *global minimum point*, or a *global minimizer*) of the function f on X. Of course, it may happen that such a point x_* doesn't exist.

Put

$$A_* = \arg\min_{x \in X} f = \{x \in X \mid f(x) = f_*\} .$$

If

$$x_0 \not\subseteq dom\ f\,, \quad f(x_0) = -\infty\,,$$

then

$$f_* = f(x_0) = -\infty\,, \quad x_0 \in A_*\,.$$

If for points $x_1 \in X$ and $x_2 \in X$ it turns out that

$$f(x_1) = -\infty\,, \quad f(x_2) = -\infty\,,$$

then we shall assume that

$$f(x_1) = f(x_2) = f_*\,.$$

Analogously, if for some point $x^* \in X$ we have $f(x^*) = f^*$, then the point x^* is called a *maximum point* (or a *global maximum point*, or a *global maximizer*) of the function f on X. Of course, it may happen that such a point x^* doesn't exist.

Put

$$A^* = \arg\max_{x \in X} f = \{x \in X \mid f(x) = f^*\} .$$

If

$$x_0 \not\subseteq dom\ f\,, \quad f(x_0) = +\infty\,,$$

then

$$f^* = f(x_0) = +\infty\,, \quad x_0 \in A^*\,.$$

If for points $x_1, x_2 \in X$ it turns out that

$$f(x_1) = +\infty\,, \quad f(x_2) = +\infty\,,$$

then we assume that $f(x_1) = f(x_2) = f^*$.

In other words, a point $x_* \in X$ is a global minimum point of f on X, if

$$f(x_* \leqslant f(x) \quad \forall x \in X\,, \tag{2.6}$$

and a point $x^* \in X$ is a global maximum point of f on X, if

$$f(x^*) \geqslant f(x) \quad \forall x \in X\,. \tag{2.7}$$

A point $x_* \in X$ is called a *strict global minimum point* or a *strict global minimizer* of the function f on X, if

$$f(x_*) < f(x) \quad \forall x \in X\,, \quad x \neq x_*. \tag{2.8}$$

A point $x^* \in X$ is called a *strict global maximum point* or a *strict global maximizer* of the function f on X, if

$$f(x^*) > f(x) \quad \forall x \in X, \quad x \neq x^* . \tag{2.9}$$

A point $x_* \in X$ is called a *local minimum point* or a *local minimizer* of f on X, if there exists a $\delta > 0$, such that

$$f(x_*) \leqslant f(x) \quad \forall x \in X : \rho(x, x_*) < \delta . \tag{2.10}$$

If $\delta = +\infty$ then the point x_* is a *global minimum point*. A point $x_* \in X$ is called a *strict local minimum point* or a *strict local minimizer* if there exists a $\delta > 0$, such that

$$f(x_*) < f(x) \quad \forall x \in X : x \neq x_*, \quad \rho(x, x_*) < \delta . \tag{2.11}$$

A point $x^* \in X$ is called a *local maximum point* or a *local maximizer* of f on X, if there exists a $\delta > 0$, such that

$$f(x^*) \geqslant f(x) \quad \forall x \in X : \rho(x, x^*) < \delta . \tag{2.12}$$

If $\delta = \infty$ then the point x^* is a *global maximum point*. A point $x^* \in X$ is called a *strict local maximum point* or a *strict local maximizer* if there exists a $\delta > 0$, such that

$$f(x^*) > f(x) \quad \forall x \in X : x \neq x^*, \quad \rho(x, x^*) < \delta . \tag{2.13}$$

If for some point $\bar{x} \in X$ we have $f(\bar{x}) = +\infty$, then by definition $\bar{x}$ is a global maximum point of the function f on X; and if $f(\bar{x}) = -\infty$, then by definition $\bar{x}$ is a global minimum point of f on X.

THEOREM 2.1. *For a point $x_* \in \operatorname{dom} f$ to be a global or local minimizer of the function f on X it is necessary that*

$$f^{\downarrow}(x_*) \geqslant 0 . \tag{2.14}$$

If

$$f^{\downarrow}(x_*) > 0 , \tag{2.15}$$

then the point x_* is a strict local minimizer of f on X.

Proof. **Necessity** follows directly from the definition. Indeed, let x_* be a local or global minimizer. Then (2.10) holds, therefore

$$f^{\downarrow}(x_*) = \liminf_{\substack{x \in X \\ x \to x_*}} \frac{f(x) - f(x_*)}{\rho(x, x_*)} \geqslant 0 .$$

Sufficiency. Let condition (2.15) be satisfied at the point x_*. We have to show that a $\delta > 0$ exists such that (2.11) holds. Assume the contrary. Let us choose a sequence $\{\delta_k\}$ such that $\delta_k \downarrow 0$. By assumption, the point x_* is not a strict local minimizer, therefore there exists an $x_k \in X$, such that

$$f(x_k) \leqslant f(x_*) , \qquad \rho(x_k, x_*) \leqslant \delta_k .$$

Hence,

$$f^{\downarrow}(x_k) = \liminf_{\substack{x \in X \\ x \to x_*}} \frac{f(x) - f(x_*)}{\rho(x, x_*)} \leqslant \liminf_{k \to \infty} \frac{f(x_k) - f(x_*)}{\rho(x_k, x_*)} \leqslant 0 ,$$

which contradicts (2.15). The sufficiency is proved. □

THEOREM 2.2. *For a point $x^* \in dom\ f$ to be a global or local maximizer of the function f on X, it is necessary that*

$$f^{\uparrow}(x^*) \leqslant 0 . \tag{2.16}$$

If

$$f^{\uparrow}(x^*) < 0 . \tag{2.17}$$

then the point x^* is a strict local maximizer of f on X.

Proof is similar to that of Theorem 2.1. □

DEFINITION 2.1. A point $x_* \in X$, satisfying condition (2.14), is called an inf-*stationary point* of the function f on X. A point $x^* \in X$, satisfying condition (2.16), is called a sup-*stationary point* of f on X.

DEFINITION 2.2 A sequence $\{x_k\}$, such that

$$x_k \in X , \quad f(x_k) \to f_* = \inf_{x \in X} f(x) ,$$

is called a *minimizing sequence* (for the function f on X).

A sequence $\{x_k\}$, such that

$$x_k \in X , \quad f(x_k) \to f^* = \sup_{x \in X} f(x) ,$$

is called a *maximizing sequence* (for the function f on X).

3. *k*-th order conditions

Let $x \in dom\ f$, $k \in 0 : \infty$. Put

$$f_k^{\downarrow}(x) = \liminf_{\substack{y \in X \\ y \to x}} \frac{f(y) - f(x)}{\rho^k(x, y)} . \tag{3.1}$$

The quantity $f_k^{\downarrow}(x)$ is called the *k-th order rate of steepest descent*.

Analogously, for $x \in dom\ f$ we define the quantity

$$f_k^{\uparrow}(x) = \limsup_{\substack{y \in X \\ y \to x}} \frac{f(y) - f(x)}{\rho^k(x, y)} . \tag{3.2}$$

The quantity $f_k^{\uparrow}(x)$, is called the *k-th order rate of steepest ascent*. It is clear, that

$$f^{\uparrow}(x) = f_1^{\downarrow}(x)\ , \quad f^{\uparrow}(x) = f_1^{\uparrow}(x)\ .$$

If $f_k^{\downarrow}(x) \in \mathbb{R}$ (i.e. it is finite), then (3.1) yields the expansion

$$f(y) = f(x) + \rho^k(x, y) f^{\downarrow}(x) + \underline{a}(\rho(x, y))\ , \tag{3.3}$$

where

$$\liminf_{y \to x} \frac{\overline{a}(\rho(x, y))}{\rho^k(x, y)} = 0\ . \tag{3.4}$$

Analogously, if $f_k^{\downarrow}(x) \in \mathbb{R}$, then (3.2) implies the expansion

$$f(y) = f(x) + \rho^k(x, y) f^{\uparrow}(x) + \overline{a}(\rho(x, y))\ , \tag{3.5}$$

where

$$\limsup_{y \to x} \frac{\overline{a}(\rho(x, y))}{\rho^k(x, y)} = 0\ . \tag{3.6}$$

The following conditions for an extremum hold.

THEOREM 3.1. *For a point $x_* \in$ dom f to be a global or local minimizer of the function f on X it is necessary that*

$$f_k^{\downarrow}(x_*) \geqslant 0 \forall k \in 0 : \infty\ . \tag{3.7}$$

If for some $k \in 0 : \infty$ it turns out that

$$f_k^{\downarrow}(x_*) > 0\ , \tag{3.8}$$

then x_* is a strict local minimizer of f on X.

Proof is similar to that of Theorem 2.1. □

THEOREM 3.2. *For a point $x^* \in$ dom f to be a global or local maximizer of the function f on X it is necessary that*

$$f_k^{\uparrow}(x^*) \leqslant 0 \forall k \in 0 : \infty\ . \tag{3.9}$$

If for some $k \in 0 : \infty$ it turns out that

$$f_k^{\uparrow}(x^*) < 0\ , \tag{3.10}$$

then x^* is a strict local maximizer of f on X.

Proof is similar to that of Theorem 2.2. □

DEFINITION 3.1. We say that a point x_0 is an inf-*stationary point of the k-th order* of the function *f*, if

$$f_i^{\downarrow}(x_0) = 0 \forall i \in 0 : k .$$

A function f is called *lower semicontinuous* at a point x_0, if

$$\liminf_{x \to x_0} f(x) = f(x_0) .$$

Clearly, if a function f is lower semicontinuous at a point x_0, then the point x_0 is an inf-stationary point of the zero order.

DEFINITION 3.2. We say that a point x_0 is a sup-*stationary point of the k-th order* of the function f, if

$$f_i^{\uparrow}(x_0) = 0 \forall i \in 0 : k .$$

A function f is called *upper semicontinuous* at a point x_0, if

$$\limsup_{x \to x_0} f(x) = f(x_0) .$$

Clearly, if a function f is upper semicontinuous at a point x_0, then the point x_0 is a sup-stationary point of the zero order.

REMARK 3.1. It is not difficult to see that the following property holds.

For a function f to be continuous at a point x_0, it is necessary and sufficient that

$$f_0^{\downarrow}(x_0) = f_0^{\uparrow}(x_0) = 0 .$$

In other words, *a function f is continuous at a point x_0 if and only if it is both upper and lower semicontinuous at this point*.

REMARK 3.2. Theorems 3.1 and 3.2 imply the following property: At any point $x \in dom\ f$ either

$$f_k^{\downarrow}(x) \geqslant 0 \forall k \in 0 : \infty ,$$

or

$$f_k^{\downarrow}(x) \leqslant 0 \forall k \in 0 : \infty .$$

If for some $k \in 0 : \infty$ we have $f_k^{\downarrow}(x) > 0$, then the point x is a strict local minimizer.

If for some $k \in 0 : \infty$ it turns out that $f_k^{\downarrow}(x) < 0$, then the point x is not a local minimizer.

Analogously, at any point $x \in dom\ f$ either

$$f_k^{\uparrow}(x) \leqslant 0 \forall k \in 0 : \infty ,$$

or

$$f_k^{\uparrow}(x) \geqslant 0 \forall \in 0 : \infty .$$

If for some $k \in 0 : \infty$ it turns out that $f_k^{\uparrow}(x) < 0$, then the point x is a strict local maximizer.

If for some $k \in 0 : \infty$ *we have* $f_k^{\uparrow}(x) > 0$, *then the point x is not a local maximizer*.

And only in the case where

$$f_k^{\downarrow}(x) = 0 \forall k \in 0 : \infty$$

or

$$f_k^{\uparrow}(x) = 0 \forall k \in 0 : \infty ,$$

we are unable to make any conclusion whether the point x is an extremum point, or not. The following examples demonstrate that in such cases any situation is possible.

In the examples below $X = \mathbb{R}$, $\rho(x, y) = |x - y|$, $x_0 = 0$.

EXAMPLE 1. Let

$$f(x) = \begin{cases} x^2 , & x \neq 0 , \\ 1, & x = 0 \end{cases}$$

It is clear that

$$f_0^{\downarrow}(x_0) = -1 , \quad f_0^{\uparrow}(x_0) = -1 , \quad f_k^{\downarrow}(x_0) = f_k^{\uparrow}(x_0) = -\infty \forall k \in 1 : \infty .$$

Thus, the sufficient condition for a maximum holds at the point $x_0 = 0$.

EXAMPLE 2. Let

$$f(x) = \begin{cases} e^{-1/|x|} , & x \neq 0 , \\ 0 , & x = 0 . \end{cases}$$

It is easy to find that

$$f_k^{\downarrow}(x_0) = f_k^{\uparrow}(x_0) = 0 \forall k \in 0 : \infty .$$

This is just the case where we are unable to get any conclusion on the extremality of the point x_0 (though in fact this point is a minimizer).

EXAMPLE 3. Let

$$f(x) = \begin{cases} -e^{-1/|x|} , & x \neq 0 , \\ 0 , & x = 0 . \end{cases}$$

Like in Example 2, it is easy to find that

$$f_k^{\downarrow}(x_0) = f_k^{\uparrow}(x_0) = 0 \forall k \in 0 : \infty .$$

And again by means of Theorems 3.1 and 3.2 we are unable to make any conclusion on the extremality of the point x_0 (though in fact this point is a local maximizer).

EXAMPLE 4. Let

$$f(x) = \begin{cases} e^{-1/x} & x > 0\,, \\ -e^{1/x}\,, & x < 0\,, \\ 0\,, & x = 0\,. \end{cases}$$

We have

$$f_k^{\downarrow}(x_0) = f_k^{\uparrow}(x_0) = 0 \forall k \in 0 : \infty\,.$$

Like in Examples 2 and 3, by means of Theorems 3.1 and 3.2 we are unable to make any conclusion on the extremality of the point x_0 (though in fact this point is neither a local maximizer, nor a local minimizer).

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